On Formal Concepts of Testability

Xavier Caicedo [1] Alvaro J. Riascos [2]
University of los Andes

July 23, 2013

[1]Department of Mathematics
[2]Faculty of Economics and Quantil

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Introduction: Motivation

- Many scientific theories are motivated and constructed using an axiomatic approach.
- In economics, decision theory is a good example.
- An axiomatic approach provides: a unifying framework, motivates generalizations, raise and link decidability and computability issues among others.
- We focus on questions regarding testability.
- Testability has long been related to specific ways of axiomatizing scientific theories (they may even be the defining characteristic of what it means to be a scientific theory).

Introduction: Example

Example (GARP)

The generalized axioms of revealed preference characterize rationalizable finite data sets and suggest that revealed preference theory is testable (any violation of the axioms in a finite data set provides an instance of falsiability).

Introduction: What we do

- This research project introduces a formal framework that allows to unify some existing proposals of:
  1. Empirical content of a theory.
  2. What it means to be testable.
- We provide a structural (algebraic) characterization of the empirical content of any axiomatizable class of structures in a compact logic.
- This characterization motivates several generalizations of the concept of empirical content and their synthactic characterizations.
- The approach is useful from a social point view. It provides a way of legitimizing proposed scientific arguments.

Introduction: Related literature

- Chambers, Ch., Echenique, F. and E. Shmaya. 2012. The Axiomatic Structure of Empirical Content.
- Simon, H and G. Groen. 1973. Ramsey Eliminability and the Testability of Scientific Theories.

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Data, Structures and Testability: Introduction

- We focus on formal definitions within first order logic.
- Although not many interesting scientific theories are formalizable within this language it is rich enough to raise several interesting questions.
- Let L be a language with no function symbols.

Data

Definition (Data Sets)

Let L' be a language with a finite number of constants and relation symbols such that $L' \subseteq L$. An L'-data set $\mathfrak{D}$ is a finite L'-structure. $\mathfrak{D} = (D, (R^{\mathfrak{D}})_{R \in L'}, (c^{\mathfrak{D}})_{c \in L'})$.

Structures

Definition (Classes of Structures)

Let $\mathfrak{T}$ be a class of L-structures.

- Structures are concrete representations (mathematical objects) meant to rationalize (to be consistent) with observable data.

Data, Structures and Testability: Examples

Example (Revealed preference theory)

Consider the language of revealed preference theory $L = (\preccurlyeq, \prec)$. An L structure is $\mathfrak{M} = (M, \preccurlyeq^{\mathfrak{M}}, \prec^{\mathfrak{M}})$. The class of structures of:

- Rational choice (weak order) is the class of structures $\mathfrak{T}_{wo}$ that are models of: completness axiom, transitive axiom and an axiom that relates $\preccurlyeq^{\mathfrak{M}}$ and $\prec^{\mathfrak{M}}$.
- Utility representations is the class of structures $\mathfrak{T}_u$, for which there is a utility function representation of $\preccurlyeq$.

- Clearly: $\mathfrak{T}_u \subset \mathfrak{T}_{wo}$.

Data, Structures and Testability: Consistency

Definition (Consistency of Data Sets)

A data set $\mathfrak{D}$ is consistent with an L-structure $\mathfrak{M}$ if there is an injective homomorphism of $\mathfrak{D}$ into $\mathfrak{M}$. We denote this by $\mathfrak{D} \rightarrow_{1-1} \mathfrak{M}$.

- Note that we require data sets to be homomorphically embedded in structures rather that isomorphically embedded.
- This allows for partial observability, a key difference with other notions of consistency (rationalizable data sets).
- $\mathfrak{T}_u$ makes more claims than $\mathfrak{T}_{wo}$. The former is consistent with less data sets than the latter.

Definition (Testability)

Let $\mathfrak{T}$ be a class of structures and $\mathfrak{M}$ any L-structure.

1. $\mathfrak{D}$ falsifies $\mathfrak{M}$ if there is no injective homomorphism of $\mathfrak{D}$ into $\mathfrak{M}$.
2. $\mathfrak{D}$ falsifies $\mathfrak{T}$ if $\mathfrak{D}$ falsifies $\mathfrak{M}$ for all $\mathfrak{M} \in \mathfrak{T}$.
3. $\mathfrak{M}$ or $\mathfrak{T}$ are falsifiable if there is some data set $\mathfrak{D}$ that falsifies $\mathfrak{M}$ or $\mathfrak{T}$.

Data, Structures and Testability: Empirical content

Definition (Empirical Content)

The empirical content $ec(\mathfrak{T})$ of theory $\mathfrak{T}$, is the class of all structures $\mathfrak{M}$ such that $\mathfrak{T}$ is not falsified by any data set $\mathfrak{D}$ consistent with $\mathfrak{M}$.

- Intuitively, the empirical content of a class of structures is the class of all structures that do not add consistent data sets that were not already consistent with the class of structures.

Example

$\mathfrak{T}_u \subsetneq ec(\mathfrak{T}_u)$ (think of lexicographic preferences).

Example

Rational choice and utility maximization are undistinguishable with finite data sets: $ec(\mathfrak{T}_{wo}) = ec(\mathfrak{T}_u)$.

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Syntactic Characterization

Definition (UNCAF Formulas)

A universal negation of a conjunction of atomic formulas (UNCAF), in the sense of Chambers et.al (2013) is a sentence of the form:

$$\forall v_1 ... \forall v_n \neg(\phi_1(v_1...v_n) \wedge ... \wedge \phi_m(v_1...v_n)) \tag{1}$$

where $\phi_i(v_1...v_n)$ are all atomic formulas with at most $v_1, ..., v_n$ as free variables or $\phi_i(v_1...v_n)$ is of the form $\neg t = s$ where t and s are terms (i.e., constants or variables within the set $\{v_1, ..., v_n\}$).

- The completness axiom is not UNCAF.

Syntactic Characterization

Definition (UNCAF Formulas of a Class of Structures)

Given a class of structures $\mathfrak{T}$, denote by $UNCAF(\mathfrak{T})$ the set of all UNCAF sentences true in every structure of $\mathfrak{T}$.

- The following is the main result in Chambers et.al (2012).

Syntactic Characterization

Theorem (Syntactic Characterization of Empirical Content)

For every class of L-strutures $\mathfrak{T}$, $ec(\mathfrak{T}) = \{\mathfrak{M} : \mathfrak{M} \models UNCAF(\mathfrak{T})\}$.

- Notice that the theorem claims that, no matter if $\mathfrak{T}$ is axiomatizable, $ec(\mathfrak{T})$ is axiomatizable.

Syntactic Characterization: Example

- Rational choice and utility maximization are undistinguishable from finite data. They have the same empirical content.
- Rational choice is not axiomatized by UNCAF formulas (think of completness axiom). Utility maximization is not first order axiomatizable.
- Their empirical content is axiomatized by UNCAF formulas (SARP).

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Structural Characterization

Theorem

If $\mathfrak{T}$ is axiomatizable in a logic that satisfies the compacteness theorem then $ec(\mathfrak{T}) = \{\mathfrak{M} : \exists \mathfrak{A} \in \mathfrak{T}, \mathfrak{M} \rightarrow_{1-1} \mathfrak{A}\}$.

- Notice we require the class of structures to be axiomatizable.
- The theorem is true in any compact logic (for example IF logic).

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Weak Empirical Content

- The structural characterization of empirical content suggests a natural generalization.

Definition (Weak Consistency of Data Sets)

A data set $\mathfrak{D}$ is (weakly) consistent with an L-structure $\mathfrak{M}$ if there is an homomorphism of $\mathfrak{D}$ into $\mathfrak{M}$. We denote this by $\mathfrak{D} \rightarrow \mathfrak{M}$.

Weak Empirical Content

Definition (Weak Empirical Content)

The weak empirical content $ec_w(\mathfrak{T})$ of theory $\mathfrak{T}$, is the class of all structures $\mathfrak{M}$ such that $\mathfrak{T}$ is not (strongly) falsified by a data set $\mathfrak{D}$ weakly consistent with $\mathfrak{M}$.

- It is easy to see that the analogous syntactic and structural characterizations are the following.

Weak Empirical Content

Theorem (Characterization of Weak Empirical Content)

For every class of L-strutures $\mathfrak{T}$:

1. $ec_w(\mathfrak{T}) = \{\mathfrak{M} : \mathfrak{M} \models UNCAF_w(\mathfrak{T})\}$, *where* $UNCAF_w(\mathfrak{T})\}$ *is the set of all universal negations of conjuntion of atomic formulas (in the standard language L) that are true in every structure of* $\mathfrak{T}$.
2. *If* $\mathfrak{T}$ *is axiomatizable in a logic that satisfies the compacteness theorem then* $ec_w(\mathfrak{T}) = \{\mathfrak{M} : \exists \mathfrak{A} \in \mathfrak{T}, \mathfrak{M} \to \mathfrak{A}\}$.
3. $ec(\mathfrak{T}) \subseteq ec_w(\mathfrak{T})$.

Strong Empirical Content

- A natural strengthening of the concept of consistency of data sets occurs when rather than homomorphic embeddings we use isomorphic embeddings.
- This leads to $ec_s(\mathfrak{T})$ and Simon et.al characterization.
- This common framework allows us to prove:

$$ec_s(\mathfrak{T}) \subseteq ec(\mathfrak{T}) \subseteq ec_w\mathfrak{T}$$